UDC 532.5

ON THE SOLUTION OF THE TWO-DIMENSIONAL PROBLEM OF GLIDING OVER A HEAVY FLUID UNDULATING SURFACE

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The problem of a two-dimensional profile sliding over the undulating surface of a perfect incompressible heavy fluid of infinite depth is solved numerically with the use of Sedov's theory [1, 2]. Forces acting on the profile and the shape of the fluid free surface are determined.

According to Sedov high speeds of motion correspond to waves of considerable length. Hence, when the profile slightly leads the waves specified ahead of it, the solution for any position of the profile on the wave contour does not greatly differ from the stationary solution. The two-dimensional problem of gliding over the surface of a fluid at rest was numerically solved in [3] by Sedov's method. The finding related to hydrodynamic forces at the profile obtained in [3] are in agreement with data in [4]. Determination of the spray filament at the leading edge makes it possible to obtain by the considered method a more exact solution.

1. Let us consider a slightly curved two-dimensional profile gliding over the surface of a perfect incompressible heavy fluid of infinite depth and density ρ, whose horizontal translational velocity c is constant. We shall use a system of coordinates attached to the profile, as shown in Fig. 1, and denote the profile projection on the fluid equilibrium level by $2a$. The profile angle of inclination to the OX - axis at any of its points is assumed small. Motion of the fluid relative to the profile is assumed to be steady and potential. As the characteristic parameters of the problem we select the velocity c and the linear dimension a, and introduce dimensionless variables by formulas

$$x' = ax, \; y' = ay, \; v' = cv, \; \varphi' = ac\varphi, \; v' = |\operatorname{grad} \varphi'|$$

where φ is the velocity potential of the fluid absolute motion and v is the absolute velocity of motion. The problem reduces to the determination of the characteristic flow function $w(z) = \varphi(x, y) + i\psi(x, y)$, where $z = x + iy$, that satisfies the following conditions.

1°. When $y < 0$ the derivatives d^2w/dz^2 and dw/dz outside the neighborhood of points $z = \pm 1$ are bounded and tend to zero when $y \to -\infty$, and the derivative dw/dz at point $z = -1$ and function w at point $z = 1$ are continuous.

2°. Ahead of the profile, at $x \to +\infty$ steady waves of the form

$$w(z) = (A_1 + iA_2)e^{-i\nu z} + \text{const}, \quad \nu = ga/c^2$$

may exist. In these formulas g is the free-fall acceleration, and A_1 and A_2 are fairly small arbitrary constants. We shall consider such waves as independent of the profile motion. In the case of motion over the surface of fluid at rest: $A_1 = A_2 = 0$.

3°. When $y = 0$ and $|x| < 1$, the condition of flow around the profile is

$$\partial\varphi / \partial y = -\beta(x)$$

4°. At the free boundary $y = 0$ and $|x| > 1$ the condition of constant pressure is

$$\partial\varphi / \partial x - \nu y = 0$$

Using Sedov's method we can express the general solution of the problem as

$$w(z) = e^{-i\nu z}\left[A_1 + iA_2 + \int_{+\infty}^{z}\left(\frac{i\gamma_2}{\sqrt{z^2-1}} + \frac{df}{dz}\right)e^{i\nu z}\,dz\right] \tag{1.1}$$

$$\gamma_2 = \sum_{n=1}^{\infty} na_n(-1)^n$$

$$f(z) = i\sum_{n=1}^{\infty} a_n\left(z - \sqrt{z^2-1}\right)^n \tag{1.2}$$

where the coefficient γ_2 represents the circulation around an equivalent wing substituted for the gliding profile and selected on the basis of the

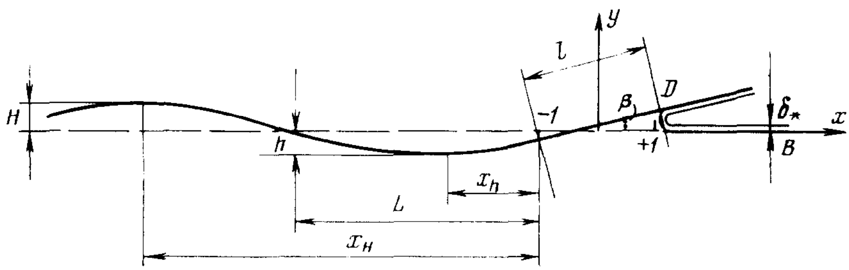

Fig. 1

Joukosky—Chaplygin condition. We have to determine in the course of solution the real coefficients a_n that satisfy the linear system of algebraic equations

$$(2k+1)\,a_{2k+1} + \frac{2\nu}{\pi}\sum_{j=0}^{\infty}\left[\frac{1}{4(j+k+1)^2-1} - \frac{1}{4(j-k)^2-1}\right]a_{2j+1} + b_{2k+1}^{*} = 0, \qquad k = 0, 1, 2, \ldots \tag{1.3}$$

$$2ka_{2k} + \frac{2\nu}{\pi}\sum_{j=1}^{\infty}\left[\frac{1}{4(j+k)^2-1} - \frac{1}{4(j-k)^2-1}\right]a_{2j} + b_{2k}^{*} = 0$$

$$k = 1, 2, 3, \ldots$$

Coefficients b_{2k+1}^{*} and $b_{2k}{}^{*}$ of this system linearly depend on parameters ν, β, γ_2, φ_1, and ψ_1, where $\varphi_1 + i\psi_1 = w(+1)$.

2. Let us consider the simplest version of the problem, when $\beta = \text{const}$ and $A_1 \neq 0$ and $A_2 \neq 0$.

The basic matrix of system (1.3) was obtained in [3] with four coefficients a_n taken into account. The rapid decrease of coefficients a_n with increasing n justifies the rejection of all subsequent terms of series (1.2). System (1.3) can be represented in the matrix form $\mathbf{A}\bar{a} = \bar{b}$, where $\mathbf{A} = \|a_{ik}\|$ is the basic fourth order matrix whose elements are defined by formulas

$$a_{11} = \frac{8}{3}\frac{\nu}{\pi} + 1 - \nu^2(P_0 + P_1) - \nu(Q_0 + Q_1) + \frac{\pi}{2}\nu$$

$$a_{12} = 2\nu^2(P_0 - P_2) + 2\nu(Q_0 - Q_1) - \pi\nu$$

$$a_{13} = -3\nu^2(P_0 + P_3) - 3\nu(Q_0 + Q_3) + \frac{3}{2}\pi\nu - \frac{8}{15}\frac{\nu}{\pi}$$

$$a_{14} = 4\nu^2(P_0 - P_4) + 4\nu(Q_0 - Q_4) - 2\pi\nu$$

$$a_{21} = -\frac{16}{9}\frac{\nu}{\pi} + \frac{1}{2}\nu^2(P_0 + P_1), \quad a_{22} = \frac{256}{45}\frac{\nu}{\pi} - \nu^2(P_0 - P_2) + 2$$

$$a_{23} = -\frac{16}{3}\frac{\nu}{\pi} + \frac{3}{2}\nu^2(P_0 + P_3), \quad a_{24} = \frac{2048}{315}\frac{\nu}{\pi} - 2\nu^2(P_0 - P_4)$$

$$a_{31} = -\frac{8}{15}\frac{\nu}{\pi}, \quad a_{32} = 0, \quad a_{33} = \frac{72}{95}\frac{\nu}{\pi} + 3, \quad a_{34} = 0$$

$$a_{41} = -\frac{32}{225}\frac{\nu}{\pi}, \quad a_{42} = -\frac{512}{1575}\frac{\nu}{\pi}, \quad a_{43} = -\frac{96}{225}\frac{\nu}{\pi}$$

$$a_{44} = \frac{4096}{1575}\frac{\nu}{\pi} + 4$$

$\bar{a} = (a_1, a_2, a_3, a_4)$ is the column of coefficients a_n which is to be determined and $\bar{b} = (b_1, b_2, b_3, b_4)$ is a column whose coefficients are determined by formulas

$$b_1 = \beta + {}^5/_8\,\nu^2\beta, \; b_2 = -{}^1/_2\,\nu^2\beta, \; b_3 = {}^1/_8\,\nu^2\beta, \; b_4 = 0$$

The basic matrix $\mathbf{A}$ remains unchanged when the independent wave is taken into account, and only its first two elements get additions in the right-hand side of system (1.3)

$$b_{1*} = b_1 + \nu(A_1 \cos\nu + A_2 \sin\nu) - \nu^2(A_2 \cos\nu - A_1 \sin\nu)$$

$$b_{2*} = b_2 + \frac{\nu^2}{2}(A_2 \cos\nu - A_1 \sin\nu)$$

where b_1 and b_2 are coefficients of the solution in the case of absence of wave.

Functions P_n and Q_n $(n = 0, 1, 2, 3, 4)$ were determined in [3,4] for small ν in the form of series in ν. The specified accuracy makes it possible to consider in these series a limited number of terms. The selection of parameters A_1 and A_2 makes it possible to determine the independent wave amplitude $A = \sqrt{A_1^2 + A_2^2}$ and, also, the position of the profile on the wave form. The lift and moment coefficients calculated for several Froude numbers in the case of $A_1 = A_2$ when the profile is at the wave crest

$$c_y = \frac{2Y}{\rho c^2 l \beta} = -\frac{\pi\gamma_2}{\beta}, \quad m_z = \frac{8M_z}{\rho c^2 l^2 \beta\pi} = \frac{a_1}{\beta}$$

are shown in Fig. 2, where curves *1*, *2* and *3* correspond to A/β equal 0.5, 0.714 and 5.0, respectively, while curve *4* represents Chaplygin's calculations ($A = 0$). The difference between the first three curves and that calculated for $A = 0$ is significant only at small Froude numbers.

The formula for calculating the ζ-ordinates of the fluid free surface behind the profile in the case of free waves is of the form

$$\zeta = A_1 \sin\nu\xi + A_2 \cos\xi\nu + \gamma_2[P_0 \cos\nu(\xi+1) - Q_0 \sin\nu(\xi+1)] -$$

$$\sum_{n=1}^{4} n a_n [P_n \cos\nu(\xi+1) - Q_n \sin\nu(\xi+1)] +$$

$$\gamma_2 \left[\frac{\pi}{2} \sin \nu (\xi + 1) - \frac{\pi}{2} \sin \nu (1 - \xi) - \right.$$

$$\left. \nu \int_{-1}^{1} \arcsin t \cos \nu (\xi + t)\, dt \right] - \sum_{n=1}^{4} a_n \left[\cos \nu (1 + \xi) - \right.$$

$$\left. \cos (\pi n - \nu + \nu\xi) + \nu \int_{-1}^{1} \sin (\nu t + \nu\xi + n \arcsin \sqrt{1 - t^2})\, dt \right] +$$

$$\gamma_2 \left[\frac{\sqrt{\xi^2 - 1}}{\xi} - \int_{1}^{\xi} \sqrt{t^2 - 1}\, \frac{\nu t \sin \nu (\xi - t) - \cos \nu (\xi - t)}{t^2}\, dt \right] -$$

$$\sum_{n=1}^{4} a_n \left[(-1)^{n-1} (\xi - \sqrt{\xi^2 - 1})^n + (-1)^n \cos \nu (\xi - 1) + \right.$$

$$\left. (-1)^n \nu \int_{1}^{\xi} (t - \sqrt{t^2 - 1})^n \sin \nu (\xi - t)\, dt \right]$$

where $x = -\xi$.

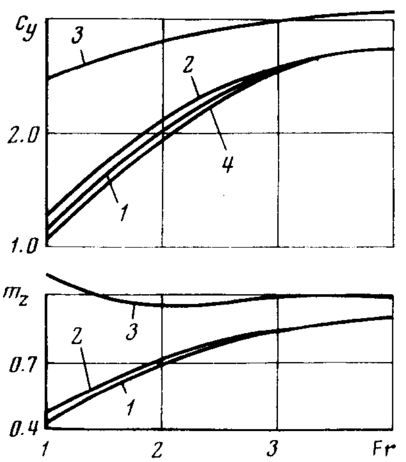

Fig. 2

Examples of the wave trough form calculated behind the profile are shown in Fig. 3, where the solid and the dash lines relate, respectively, to $A/\beta = 0.5$ and $A / \beta = 5.0$ with $\beta = 0.02$, and are given for several Froude numbers. As in the case of $A = 0$, the plate trailing edge is always above the deepest spot of the trough and below the highest spot of the subsequent rise of fluid in the trough. The fluid surface form is that of a $2\pi/\nu$ -long sine wave at virtually one wave length behind the profile. Height of that wave is determined directly. For a quantitative determination of the form of the fluid free surface it is convenient to use its horizontal and vertical dimensions denoted, respectively, by x_h, L and x_H, and by h and H (see Fig. 1). In the considered example an increase of A results in the shift of maximum trough depth away from the profile. In the beginning this decreases the trough depth. However a further increase of parameter A shows that the closer the independent wave trough is to the deepest point of the generated trough, the deeper is the latter. Such resonance of the independent wave and of the trough is clearly seen when $A / \beta = 5.0$ where the trough is even deeper than for $A = 0$. It should be noted that for an excessively high wave the results obtained by the theory used here become less accurate. The calculation results vary periodically depending on the relative position of the profile over the wave. All other possible positions of the profile relative to the independent wave are considered below on a somewhat different basis. If the wave is assumed fairly long in comparison with the profile wetted length, it is possible to consider a certain lead of the profile over the form of that wave, and determine all possible positions of the profile over it.

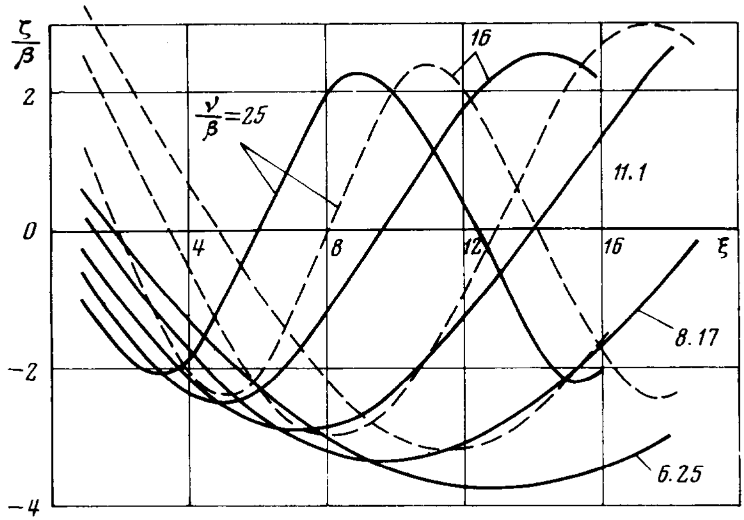

Fig. 3

3. Small ν correspond to long waves. For example, when $\nu < 0.5$, which corresponds to the Froude number $\mathrm{Fr} = (2\nu)^{-1/2}$ (relative to $l = 2a$) , the relative wave length is $\lambda > 12.5$. Since we use the linear theory, it is possible to represent the motion of the profile over a long wave with a small lead over the latter as a sequence of a number of its positions relative to the wave in the stationary problem. Let us consider such quasi-stationary motion on the example of a wave specified amplitude.

The general solution with allowance for independent waves yields for the form of the free fluid surface at considerable distance ahead of the profile an equation of the form

$$\psi(x, 0) = -A_1 \sin \nu x + A_2 \cos \nu x \qquad (3.1)$$

The lead of the profile over independent waves can be defined with the use of some supplementary parameter μ , and (3.1) can be represented as follows:

$$\psi_*(x, 0) = -A_1^* \sin \nu x + A_2^* \cos \nu x$$
$$A_1^* = A_1 \cos \mu - A_2 \sin \mu, \quad A_2^* = A_1 \sin \mu + A_2 \cos \mu$$

We may assume that $\mu = Vt / a$, where V is the lead velocity and t is the time. The solution for any arbitrary position of the profile over a wave of specified amplitude is then determined by solving the stationary problem for various A_1^* and A_2^* , for which it is sufficient to take μ from the interval $(0, 2\pi)$. Examples of calculations of the free surface form that correspond to the deepest and shallowest wave trough behind the profile (see Fig. 4). The trough depth can vary by $2A / \beta$. The smallest depth obtains when $\mu = 0.715$ and tne greatest for $\mu = 1.715$.

This result has a reasonably sound physical basis. Thus, when $\mu = 0.715$ at point $x = -1$, which determines the trailing edge of the profile, the function that defines the wave form is zero and its derivative is positive. Thus the rise in the independent wave in the trailing edge region in this case tends to reduce the trough depth. The lengthwise characteristics of the fluid free surface remain unchanged. The curves in Fig. 5 show that the forces and the moment at the profile vary periodically.

The curves in Figs. 4 and 5 have been calculated for $A / \beta = 0.143$ (the solid and dash lines relate, respectively, to $\nu / \beta = 4.57$ and $\nu / \beta = 7.15$, with $\beta = 0.07$).

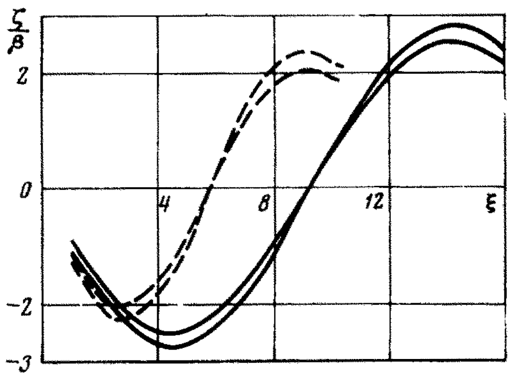

Fig. 4

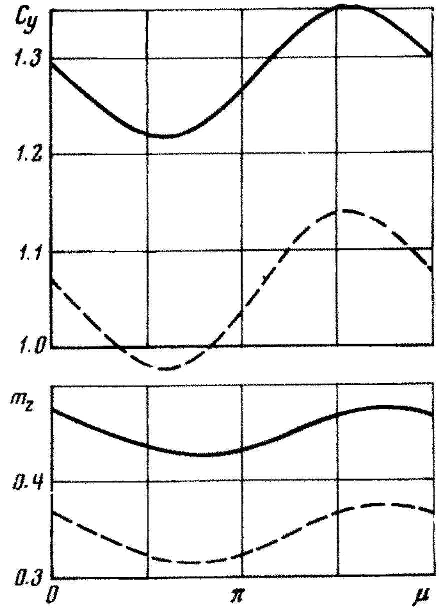

Fig. 5

4. The theory on which these calculations are based does not yield a correct solution in the neighborhood of the leading edge. This is the consequence of the known assumption that the fluid velocity in that neighborhood is low. The presence of a spray filament, similar to the infinite velocity at the leading edge of a thin wing, is characteristic of that neighborhood. The method used below makes it possible to determine the spray filament thickness and the exact boundary of the free surface at infinite distance ahead of the profile, and also to define more accurately that filament in the region of its base. The theory similar to that of the theory of wing and the theory of filaments for a weightless fluid yield the same results, except in the small neighborhood $\sim \beta^2$ of the leading edge. Such filament was determined in [5] in the case of weightless fluid, where it was shown that the solution of the filament problem in the region of the stagnation point is of the form $w_c = 4ci\sqrt{\delta z / \pi}$, while the solution by the method considered here is in that region of the form $w_a = id\sqrt{z}$, where d = const. The condition of smooth transition of the main stream to the region of the spray filament at distance $\sim \beta^2$ from the stagnation point yields for the filament thickness the formula

$$\delta = \pi d^2 / 16 c^2 \qquad (4.1)$$

The filament is readily determined by the introduction of the so-called wetted length l (see Fig. 1) on which virtually the total pressure is concentrated. For small angles of travel the formula $l = 4\delta / (\pi\beta^2)$ is valid. This yields the known relationship for a weightless fluid: $\delta_* = \pi\beta^2/2$, where $\delta_* = \delta/2a$. As shown by Sedov, for a ponderable fluid the analogy between gliding and the motion of a wing holds, except in the neighborhood of the leading edge. Assuming that a formula of the form (4.1) is also valid for a ponderable fluid, we have $\delta_* = \pi d^2{}_0 / 4$ and for the determination of the constant d_0

$$\frac{dw}{dz} = \frac{df}{dz} + \frac{i\gamma_2}{\sqrt{z^2-1}} - i\nu w \sim [2(z-1)]^{-1/2} i \sum_{n=1}^{4} a_n n [(-1)^n - 1]$$

From the last formula $d_0 = -\sqrt{2}\,(a_1 + 3a_3)$ and for the filament thickness in the stated approximation we have $\delta_* = {}^1/_2\,\pi\,(a_1+3a_3)^2$. When $\nu \to 0$, $a_1 \to \beta$ and $a_n \to 0\,(n = 2,3,4)$. This result makes possible a more exact definition of flow in the leading edge region. According to the exact theory the streamlines that determine the ordinates of the fluid free surface behind and ahead of the profile differ by the filament thickness. The free boundary of fluid behind the profile is determined in the case of the equivalent wing, and in that of the gliding plate by the streamline which reaches the stagnation point. Since the wing chord $l = 2a$, hence for $\delta_* = 0$ the boundary of the fluid free surface ahead of the profile is defined by curve BD (see Fig. 1). As noted in [3], that curve in the vicinity of the leading edge is nearly vertical. The latter follows from the assumption that the leading edge can be reached by moving continuously over the free surface ahead of the profile, which in reality is not so.

In concluding we note that the analysis presented here completes the numerical investigation of the two-dimensional problem which was started in [4] and [3] and was based on its aerodynamic analogy. Results obtained here are valid for a rectilinear profile. It is established that the effect of regular perturbations of fluid at considerable distance ahead of the profile on the forces acting on it is significant only at moderate and low Froude numbers. At considerable Froude numbers the problem, as formulated here, involves a long wave for which the obtained results are valid with known accuracy also in the case of a small lead of the profile over waves specified ahead of it. The forces and moments acting at the profile are periodic of period $2\,\pi c\,/\,(\nu V)$. Ordinates of the fluid free surface are determined by simple superposition. The maximum height of the initial wave rise from the trough over the depth of the wave trough behind the profile is equal to the height of the oncoming wave. The longitudinal pattern of the boundary of the fluid free surface remains unchanged. To improve the accuracy of solution derived by the described theory in the leading edge region it was assumed that it is analytic in that region and similar to that obtained in the theory of weightless fluid. The joining of the two solutions at the filament base makes it possible to take into account the filament ponderability, and improve the accuracy of results obtained by the considered theory. Passing to limit at considerable Froude numbers yields for the filament thickness the same result as in the case of weightless fluid.

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